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HAMMING DISTANCES AND GENERALIZED HAMMING GRAPH

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Abstract: In this paper we first highlight applications of Hamming distance in allied areas and further keeping applications of Hamming distance in mind we introduce the notion of Generalized Hamming graph which is generalization of the notion of Hamming graph in sense of Hamming distances. Besides this, we have studied the concept of labeling in the realm of embedding of arbitrary graph into generalized Hamming graph. Several new problems for further research are also indicated.

Keywords and Phrases: Graph labeling, Code, Hamming graph, Hamming distance, String.

2020 Mathematics Subject Classification: 05C99, 05C78.

1. Introduction

Throughout the paper the considered graph are simple. We assume that the reader is familiar with basic notions of graph theory which can be found in any standard text books for notations no longer especially noted or described here we refer the reader to the textbook [6]. Similarly, we assume familiarity with the basic concepts of linear algebra but perhaps not as much as graph theory. This work is basically relay on the blending of both above mentioned area's.

In this work, we use the set $S := \mathbb{Z}_2^d = \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{d-times} = \{(x_1, x_2, x_3, \ldots, x_d) : x_1, x_2, \ldots, x_d \in \mathbb{Z}_2\}$. For the q-tuple $\{(x_1, x_2, x_3, \ldots, x_q) : x_1, x_2, \ldots, x_q \in \mathbb{Z}_2\}$ the elements of standard basis are denoted by e_i, $1 \leq i \leq q$ and defined as $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, and $e_q = (0, 0, \ldots, 1)$. The zero vector is denoted by e_0 and is defined as $e_0 = (0, 0, 0, \ldots, 0)$.

The credit of original work in direction of coding theory goes to Richard W. Hamming [5] and the *Hamming scheme* became popular after this. According to Hamming scheme, the elements are vectors of length d on some alphabet of size q. To moving step forward, it is necessary to recall the following terminology:

Definition 1.1. *Let $x \in \mathbb{Z}_2^n$. Then the weight of vector x is denoted by $wt(x)$, and is given by $wt(x) = d(x, 0)$ (i.e., the distance of x from the zero vector).*

Equivalently, the number of 1's in the string $x = x_1x_2 \ldots x_n$ is known as the weight of x. The weight may be thought of as the magnitude of the vector.

Example 1.2. Let us take two vectors $x = (11101)$ and $y = (10110)$. Then in view of Definition 1.1, $wt(x) = 4$ and $wt(y) = 3$.

Remark 1.3. *It may be notice that two vectors of different length can have the same weight; as for instance $wt(100100) = 2$ and $wt(101) = 2$.*

Let x and y be two elements of $\mathbb{Z}_2^d$, where $x = x_1x_2 \ldots x_d$ and $y = y_1y_2 \ldots y_d$. Then the sum $x + y$ is calculated by adding the corresponding tuples of x and y under addition modulo 2.

Definition 1.4. [5] *Let $x = x_1x_2 \ldots x_m$ and $y = y_1y_2 \ldots y_m$ be two strings of length m. The Hamming distance denoted by $H_d(x, y)$ between the strings x and y is the number of i's such that $x_i \neq y_i$, $1 \leq i \leq m$. Thus $H_d(x, y)$ = the number of places at which x and y differ $= wt(x + y)$.*

Thus, the Hamming distance between two vectors is the number of bits we must change in one vector to get the other vector. The following example illustrate the concept of Hamming distance:

Example 1.5. Let $x = (1110)$ and $y = (1011)$. Then $x + y = (0101)$ and $H_d(x, y) = 2 = wt(x + y)$.

Remark 1.6. *Let B be the set of standard basis of q-tuples. Then, immediately it can be noticed that any two distinct elements in B differ with exactly two coordinates. In other words, if u and v, $u \neq v$ both belongs to B, then $H_d(u, v) = 2$.*

It is worthwhile to note here that there does not exist three vectors in $\mathbb{Z}_2^d$ mutually possessing Hamming distance 1. Now we focus our attention to Hamming graph which revolves around the Hamming distance.

Definition 1.7. [5] *Let S be a set of q elements and d be a positive integer. The Hamming graph denoted by $H(d, q)$ has vertex set S^d, the set of ordered d-tuples of elements of S and two vertices u and v are adjacent if $H_d(u, v) = 1$.*

Example 1.8. For $d = 1$, the Hamming graph $H(1, q)$ is isomorphic to complete graph K_q. On the other hand, if $q = 1$, then $H(d, 1)$ is singleton graph K_1. However, the role of d is much more important as if $d = 2$, then $H(2, q)$ is isomorphic to $q \times q$ grid.

Hamming graphs represent a family of graphs that has arisen from coding theory. There has been an extensive literature on Hamming graphs, for more details see [3, 4, 7, 8, 10].

Several applications, such as pattern recognition, multi-media and intelligent processing, require considerable memory access and data processing time. To tackle such problems, digital associative memories based on Hamming distance is required that involve the computation of Hamming weights or Hamming distance. Keeping these issues in mind several authors established many application of Hamming distance in the various field of interest some of them are as follows: making use of Hamming distance for determining the document similarity Jarrous and Pinkas established protocols in [2]. Kambs et al. [9] computed the Hamming distance d between DNA oligonucleotides using some experiments. Motivated by large-scale multimedia applications Norouzi et al. [11] derived a framework for Hamming distance metric learning, which basically includes use of specified mapping from the input space onto binary codes and it accommodates different classes of hash functions.

Due to vast literature on the concept of Hamming distance, it is not possible to mention all the applications here. In fact labeling technique is one of them, so now we restrict our attention on embedding problem through labeling. Therefore, in this paper an attempt have been made to establish the application of Hamming distance in labeling. The main motivation behind to introduce the notion of Generalized Hamming graph is to establish a connection between Hamming distance and labeling. To do this first we introduce the notion of Generalized Hamming graph and later we have shown that every graph can be embedded into Generalized Hamming graph. Several new results related to embedding of graph have also been obtained.

2. Generalized Hamming Graph

In this section we introduce new notion of graph, viz., Generalized Hamming graph denoted by $H_k(d, q)$, which is just an extension of notion of Hamming graph in sense of Hamming distances. The formal definition is as follows:

Definition 2.1. *Let S be a finite set with q elements and k, d $(k \leq d)$ be two positive integers. Let S^d denotes the set of ordered d-tuples of elements of S. The Generalized Hamming Graph denoted by $H_k(d, q)$ has vertex set S^d and any two vertices are adjacent if and only if they differ precisely in k-coordinates.*

Consequently, if $k = 1$, then $H_k(d, q)$ is isomorphic to the standard Hamming graph $H(d, q)$. The following examples depict the structure of $H_k(d, q)$ for different values of k, d and q.

Example 2.2. If $k = d = 1$, then the Hamming graph $H_1(1, q)$ is isomorphic to complete graph K_q. On the other hand, if $q = 1$ with $k = 1$, then $H_1(d, 1)$ is singleton graph K_1.

Example 2.3. If $k = 1$ and $d = 2$, then $H_1(2, q)$ is $q \times q$ grid and if $k = 2$ and $d = 2$, then $H_2(2, q)$ is 2-copies of K_2. Further if we take $q = 2$, $k = 1$ and $d = 3$, then $H_1(3, 2)$ is isomorphic to hypercube Q_3. However if $q = 2$, $k = 2$ and $d = 3$, then the Hamming graph $H_2(3, 2)$ is disconnected and is isomorphic to 2-copies of K_4. Furthermore if $q = 2$, $k = 3$ and $d = 3$, then the Hamming graph $H_3(3, 2)$ is disconnected and is isomorphic to 4-copies of K_2.

Since we are interested in Hamming graph over $\mathbb{F}_q := \mathbb{Z}_2$, so here onwards we shall calculate each parameter with respect to $q = 2$. The following result provides the formula for the total number of edges in $H_k(d, 2)$.

Theorem 2.4. *The number of edges in $H_k(d, 2)$ is given by $\binom{d}{k}\frac{q^d}{2}$.*
Proof. Consider the Generalized Hamming graph $H_k(d, 2)$. Clearly there are 2^d vertices and they themselves are d-tuples. Let $u, v \in \mathbb{Z}_2^d$ be two vertices of $H_k(d, 2)$, then u and v are adjacent if and only if $H_k(u, v) = k$. Let $u = (u_1, u_2, \ldots, u_d)$ and $v = (v_1, v_2, \ldots, v_d)$, where $u_i, v_i \in \mathbb{Z}_2$ for all $1 \leq i \leq d$. One may observe that u and v can differ at k places if and only if u_i is not equal to v_i precisely at k places. Thus in order to differ the vertex u with v at k tuples among d tuples there are $\binom{d}{k}$ choices. This implies that each vertex in $H_k(d, 2)$ is adjacent to $\binom{d}{k}$ vertices. Thus by degree theorem [6, Theorem 2.1, p. 141], one found that the number of edges in $H_k(d, 2)$ is given by $2^{d-1} \times^d C_k$.

Invoking the Theorem 2.4 the following result is straight forward.

Theorem 2.5. *The Generalized Hamming graph $H_k(d, 2)$ is $\binom{d}{k}$-regular.*

In view of Example 2.3 it is worthwhile to note that the Generalized Hamming Graph $H_k(d, q)$ need not be connected.

Remark 2.6. *It is also interesting to know that the structure of $H_k(d, q)$ vary according to the different values of k, $k \leq d$. As if we take $k = 1$, then $H_1(d, q)$ is connected, however if k increases and very close to d, then it does not happen so.*

This indicates that k and d are mutually dependent. In fact for $k = d$ or $k = d-1$, $H_k(d, q)$ is disconnected.

Motivated from the study of existing literature on Hamming graphs and applications of Hamming distance, Acharya and Pranjali [1] introduced the concept of Hamming distance k-labeling of graphs. The point of view of their paper, however, was mainly concern to labeling. With this context, in this paper we extend a systematic study of Generalized Hamming graph and affix it with the concept of Hamming distance k-labeling of graphs. Viewed in this way, the Generalized Hamming graph turns out to be a maximal graph with respect to an injective Hamming distance k-labeling of any of its spanning subgraphs. We also initiate a study of optimal injective Hamming distance k-labeling of a given graph and further we have identified certain conditions which guarantee a graph to be embedded into standard hamming graph $H_1(d, q)$.

The formal definition defined in [1] is as follows:

Definition 2.7. [1] *Let $G := (V, E)$ be a graph with vertex set $V := V(G)$ and edge set $E := E(G)$. Let H_d denotes the Hamming distance of G. A one-one function $f : V \to \mathbb{Z}_2^m$ is called the Hamming distance k-labeling of G if for each $(u, v) \in E(G)$, $H_d(f(u), f(v)) = k$.*

Example 2.8. Consider Q_3 and take $S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The Hamming distance 1-labeling of Q_3 is shown in Figure 1.

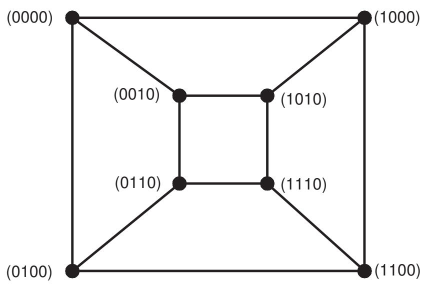

Figure 1: Hamming distance 1-labeling of Q_3

Example 2.9. Consider K_4, to label the vertices of graph take a set $S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The Hamming distance 2-labeling of K_4 is shown in Figure 2.

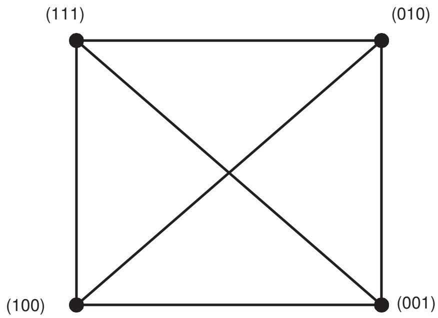

Figure 2: Hamming distance 2-labeling of K_4

Indeed it is obvious to recover the following basic facts about Hamming distance k-labeling defined as above.

Remark 2.10. *Let G be graph and H be a subgraph of G. If f is a Hamming distance k-labeling of a graph G, then f is a Hamming distance k-labeling of H as well.*

Remark 2.11. *If $f : V(G) \to \mathbb{Z}_2^m$ is Hamming distance $k-$ labeling of a graph G, then f is Hamming distance k-labeling of each supergraph G' of G, which is obtained by adding edges that join pairs of distinct points x, y that had been now no longer joined in G and having the property that $H_d(f(x), f(y)) = k$. Hence, such a maximal supergraph M_H^f so obtained is an induced subgraph of the Generalized Hamming graph of $\mathbb{Z}_2^m$. Further, if f is surjective, then $M_H^f = H_k(d, 2)$.*

It is evident from [1] the every graph does not admit the Hamming distance k-labeling for all k.

3. Main Results

This section is devoted to study of embedding of arbitrary graph into Generalized Hamming graph $H_k(d, 2)$.

Theorem 3.1. *Every finite graph of order n can be embedded into $H_2(n, 2)$.*

Proof. In order to prove the desired result, it suffices to show that complete graph admits the Hamming distance 2-labeling. To do this, define a function $f : V(K_n) \to \mathbb{Z}_2^n$ such that

$$f(v_i) = e_i, \quad 1 \leq i \leq n,$$

where e_i are the standard basis with n-tuples. Clearly, f is injective and in view of Remark 1.6, it is easy to find that for each pair of vertices $(u, v) \in E(K_n)$, $H_d(f(u), f(v)) = H_d(e_i, e_j) = 2$. This indicates that for each pair of adjacent vertices the Hamming distance between vertex labels is 2. Thus, K_n can be obtained as a subgraph of $H_2(n, 2)$ and hence it can be embedded into $H_2(n, 2)$.

One can digging out the depth of above result that mentioned f of the kind specified above nevertheless has a universality include in the fact that incidences of the vertices to the edges in the given graph G does not have impact at all, in the sense that only requirement is that G is a subgraph of $H_2(d, 2)$. This insists us to define 'optimal' Hamming distance-2 labeling of a graph.

Definition 3.2. *Let G be a finite graph of order n. The Hamming index of G, denoted by $H^{\Theta}(G)$, is the least positive integer h such that there is a set with 2^d elements with respect to which G admits a Hamming distance-2 labeling. The Hamming distance-2 labeling f of G is optimal if it uses the vertex labels of vector space $\mathbb{Z}_2^d$.*

Thus it can easily be observed that Hamming distance-2 labeling of G is just an embedding of G into $H_2(d, 2)$ and $H^{\Theta}(G)$, is the order of the smallest such Generalized Hamming-2 graph. Therefore, towards finding $H^{\Theta}(G)$ we need to pay attention to find smallest d for which G can be embedded into $H_2(d, 2)$.

Invoking Theorem 3.1 along with Definition 3.2, it follows that for a finite graph G of order n, $n \geq 2$,

$$2^{d_1} \leq H^{\Theta}(G) \leq 2^n, \tag{1}$$

where d_1 denotes the ceiling of $\log_2 n$.

The bounds in (1) immediately allows us to raise the following open problem:

Problem 3.1. *Characterize graphs for which the bounds in equation (1) are attained. Also, find the positive integer d_1 which provides optimal Hamming distance-2 labeling for these graphs.*

The concept of optimal Hamming index have been illustrated by the following example:

Example 3.3. Let us consider cycle C_4. Then by labeling technique it is found that it can be embedded into each of $H_2(6, 2)$ $H_2(4, 2)$, and $H_2(3, 2)$ but not in $H_2(2, 2)$. Therefore, the smallest order of Generalized Hamming graph is 8 and precisely the graph is $H_2(3, 2)$. Hence the optimal Hamming index $H^{\Theta}(C_4)$ is 8.

Towards looking the bounds in equation (1) it can also be ascertain that for cycle C_4, the upper bound is 16 though $H^{\Theta}(C_4)$ is 8. This indicates the presence of such graphs for which bounds are not sharp, so one can put attention to determine such structures.

The next result is quiet interesting and strengthened in sense of embedding of G in $H_k(d,2)$ for each even k; which a more general form of above stated result.

Theorem 3.4. *Every finite graph of order n can be embedded into $H_k(\frac{k}{2}n,2)$, for each even k.*

Proof. In the light of Theorem 3.1, we know that every graph admits the Hamming distance 2-labeling, the proof can be completed by using the method of extending the coordinates, but the technique is little bit different. We shall increase the length of each string by taking some copies of the original string. Suppose we extend the length of string by taking only one copy, then the original vertex label $f(v_i) \in \mathbb{Z}_2^n$ becomes the new label $f'(v_i) \in \mathbb{Z}_2^{2n}$, that is $f(v_i) = (1,0,0,\ldots,0)$ of length n has converted into the string $f^{'}(v_i) = (1,0,0,\ldots,0,1,0,0,\ldots,0)$ of length $2n$. In this way, the weight always increased with magnitude 2. Thus,

$$\begin{aligned} wt(f^{'}(v_i)+f^{'}(v_j)) &= wt(f(v_i)+f(v_j))+2 \\ &\Rightarrow H_d(f^{'}(v_i),f^{'}(v_j)) = H_d(f(v_i),f(v_j))+2 \\ &= 4, \end{aligned}$$

Proceeding in the similar vein by taking some copies of string $f^{'}$ and get the functions f'', f''', ..., $f^{''''(k_1-1)-times}$ in such a way that every time the weight of the sum of vertex labels is increased with magnitude 2 from the previous one. Thus, for the functions f'', f''', ..., $f^{''''(k_1-1)-times}$ we have

$$\begin{aligned} wt(f^{''''(k_1-1)-times}(v_i)+f^{''''(k_1-1)-times}(v_j)) &= wt(f(v_i)+f(v_j))+\sum_{1}^{k_1-1}2 \\ &= 2+2(k_1-1) \\ \Rightarrow H_d(f^{''''(k_1-1)-times}(v_i),f^{''''(k_1-1)-times}(v_j)) &= 2k_1 \end{aligned}$$

which is even. Therefore at each stage there exist a function by which we found that G can be embedded into each of $H_2(n,2)$, and $H_4(2n,2)$, $H_6(3n,2),\ldots,H_k(\frac{k}{2}n,2)$, k is even. Hence for each even k, finite graph G can be embedded into $H_k(\frac{k}{2}n,2)$.

Now the above result opens new horizon of embedding of G into $H_k(\frac{k}{2}n,2)$, for each even k. There is a curious fact about even Hamming distance, which is worth mentioning. Thus the Generalized Hamming graph $H_k(d,2)$ in which H_d is even for each pair of adjacent vertices deserves more attention. Furthermore, the above defined optimal Hamming index is canonical in the sense that we can introduce the number of optimal indices depending upon the embedding of G into $H_k(\frac{k}{2}n,2)$ with specified k. Thus we can now define the Hamming index more precisely as follows:

Definition 3.5. *Let G be a finite graph of order n and k be even positive integer. The Hamming k-index of G, denoted by $H_k^{\Theta}(G)$, is the least positive integer h such that there is a set consisting of 2^d elements with respect to which G admits a Hamming distance-k labeling. The Hamming distance k-labeling f of G is optimal if it uses the vertex labels of vector space $\mathbb{Z}_2^d$.*

From the proof of Theorem 3.4 it can easily be pointed out that if G is a graph of order n, then $H_2^{\Theta}(G) \leq 2^n$, $H_4^{\Theta}(G) \leq 2^{2n}$, $H_6^{\Theta}(G) \leq 2^{3n}$, and so on $H_k^{\Theta}(G) \leq 2^{\frac{k}{2}n}$, where k is even.

In contrast of Theorem 3.4 along with equation (1), the more precise form of bound of $H_k^{\Theta}(G)$ for a finite graph G of order $n \geq 2$, is as follows:

$$2^{d_1} \leq H_k^{\Theta}(G) \leq 2^{\frac{k}{2}n}, \tag{2}$$

where d_1 is the ceiling of $\log_2 n$.

Furthermore, the above bounds in equation (2) can raises the similar open problem as Problem 3.1, so one may attempt to tackle this more general problem. Besides this, it is also of interest to know the conditions on k which guarantee finite graph G can be properly embedded into $H_k(d, 2)$ as they lead to very interesting interplay between optimal indices, namely, $H_2^{\Theta}(G)$, $H_4^{\Theta}(G), \ldots, H_k^{\Theta}(G)$, and brings out some interesting combinatorial properties of Generalized Hamming graph.

Turning to the standard Hamming graph, i.e., $H_1(d, 2)$. it is natural to have a curiosity to know, whether every graph can be embedded into Hamming graph? A quick answer is No! as for the instance; consider complete graph K_3, then there does not exist function $f : V(K_3) \to \mathbb{Z}_2^n$ such that it can be embedded into $H_1(d, 2)$.

In fact the following problem raised in [1] is still open:

Problem 3.2. *Characterize the graphs which admit the Hamming distance 1-labeling?*

Though the above problem was solved partially in [1] for some well-known graphs. one may summarize the results as follows:

Theorem 3.6. [1] *The following graphs can be embedded into $H_1(d, 2)$.*

- *Cycle graph C_n, for even n,*
- *Every star graph $K_{1,n}$,*
- *Complete graph K_2,*
- *Every hypercube Q_n.*

The following result derived in [1], provides the necessary condition for a graph to be embedded into $H_1(d, 2)$.

Lemma 3.7. *If a finite graph G has an odd cycle, then it can not be embedded into $H_1(d, 2)$.*

At this stage, It is not difficult to see from the above lemma that bipartite-ness of a graph is the necessary condition to be embedded into $H_1(d, 2)$. This raises the following fundamental question:

Problem 3.3. *Characterize the bipartite graphs which can be embedded into $H_1(d, 2)$.*

Although it is clear from the above discussion that every graph can be embedded into $H_2(d, 2)$. But, there is still inquisitive to know what we can say about embedding of G into $H_k(d, 2)$, where k is odd and $(k \geq 3)$. In this regard, we have established the following theorem.

Theorem 3.8. *Let G be a graph embedded into $H_1(d, 2)$. Then it can also be embedded into $H_k(d, 2)$ $(k > 1)$.*

Proof. We will show the desired result, with the help of the method of extending the length of string. Consider the finite graph G which can be embedded into $H_1(d, 2)$ this indicates the existence of a function $f : V(G) \to \mathbb{Z}_2^m$ such that for each $(v_i, v_j) \in E(G)$, $H_d(f(v_i), f(v_j)) = 1$, where $f(v_i)$ and $f(v_j)$ are the strings of length of m. Now we shall extend the length of each string by one coordinate putting either 0 or 1 at $(m+1)^{th}$ place, by which the new vertex label becomes the string of length $m+1$ say $f'(v_i) \in \mathbb{Z}_2^{m+1}$ with the condition that for each adjacent pair $(v_i, v_j) \in E(G)$, $f^{'}(v_i) = (, , , \ldots, 0)$ and $f^{'}(v_j) = (, , , \ldots, 1)$. Now it can easily be seen that the new function $f^{'} : V(G) \to \mathbb{Z}_2^{m+1}$ is injective and the weight of $f^{'}(v_i) + f'(v_j)$ is increased with magnitude 1, that is,

$$\begin{aligned} wt(f^{'}(v_i) + f^{'}(v_j)) &= wt(f(v_i) + f(v_j)) + 1 \\ &\Rightarrow H_d(f^{'}(v_i), f^{'}(v_j)) = H_d(f(v_i), f(v_j)) + 1 \\ &= 2, \end{aligned}$$

which shows that G admits the Hamming distance 2-labeling.

Repeating the above procedure to add as one extra bits (either 0 or 1), we get a sequence of functions $f^{''}$, $f^{'''}$, $\ldots$, $f^{''''(k-1)-times}$ in which every time the weight of the sum of vertex labels is increased with magnitude 1 from the previous one. Thus, for the functions $f^{''}$, $f^{'''}$, $\ldots$, $f^{''''(k-1)-times}$ we have

$$
\begin{aligned}
wt(f^{''''(k-1)-times}(v_i) + f^{''''(k-1)-times}(v_j)) &= wt(f(v_i) + f(v_j)) + \sum_{1}^{k-1} 1 \\
&= 1 + (k-1) \\
\Rightarrow H_d(f^{''''(k-1)-times}(v_i), f^{''''(k-1)-times}(v_j)) &= k,
\end{aligned}
$$

Therefore for each k, $(k > 1)$ the graph G admits the Hamming distance k-labeling. Hence G can also be embedded into each of $H_k(d, 2)$.

From the rigorous analysis, one can interestingly notice that if a graph G is embedded into $H_1(d, 2)$, then it is also embedded into $H_k(d, 2)$ $(k > 1)$, i.e., embedding of G into $H_1(d, 2)$, implies embedding of G into $H_k(d, 2)$. In other words if G is subgraph of $H_1(d, 2)$, then it is a subgraph of each of $H_k(d, 2)$.

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